WILSON LOOP FOR SU(N)-GLUODYNAMICS

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Abstract

Wilson loop of SU(N) Yang–Mills theory is evaluated in the Fock–Schwinger gauge at the confined phase. The latter is characterized by presence of the singlet projector of the group of gauge transformations at infinity in any correlation function. Obtained values of the string tension and critical temperature are in agreement with lattice data.

Wilson loop [1] is being serving as confinement criterion for decades in investigations of nonabelian gauge theories. Its application was especially successful in the case of lattice simulations [2]. The approach I would like to suggest in the present report is however somewhat different from those earlier used. Our calculation relies heavily on the fact that the system is indeed in the confined phase. Namely we keep singlet projector throughout derivation. So confinement in this algebraic sense is supposed to be already proved by other means.

In the recent papers by the author in collaboration with N. A. Sveshnikov [3, 4, 5] it was demonstrated how singlet projector P_s arises inside any localized variables' Gibbs mean value. This projector of the group of gauge transformations at infinity $G_\infty = G/G_p$ provides adequate comprehension of confinement phenomenon from purely theoretical point of view. Therefore one may ask the following reasonable question. Does this projector lead to the area law in the Wilson loop? The answer turns out positive as it will be seen later on.

We prefer to work in 3-dimensional Fock–Schwinger (FS) gauge [6-8]

$$\mathbf{x}\mathbf{A}(t,\mathbf{x}) = 0\,. \tag{1}$$

Let us consider the Wilson loop

$$\mathcal{W}[\Gamma] = \langle \mathrm{Tr}\,\mathrm{P}\,\exp(g\oint_\Gamma A_\mu dx^\mu)\rangle, \quad A_\mu = A_\mu^a t^a, \quad (t^a)^\dagger = -t^a\,, \tag{2}$$

that is a gauge–invariant order parameter. It is necessary to relate it with the physical variables of the FS gauge to begin with. The zero component A_0 appears in the first order formalism as a Lagrange multiplier

$$\mathcal{L} = \dot{\mathbf{A}}\mathbf{E} - \frac{1}{2}((\mathbf{E})^2 + (\mathbf{B})^2) + A_0 \nabla \mathbf{E}\,. \tag{3}$$

This variable is determined from the equation of motion

$$E_k^a = \partial_0 A_k^a - \partial_k A_0^a + g t^{abc} A_k^b A_0^c \tag{4}$$

while the task on conditional extremum is formulated. From the (4) we get after multiplication by $\hat{x}_k$

$$A_0(t, \mathbf{x}) = - \int_z^x dy \, E_\parallel (t, y\hat{\mathbf{x}}) \, . \tag{5}$$

The lowest integral limit z is an arbitrary point, nevertheless such ambiguity disappears in the integral over closed circuit.

The most suitable contour Γ consists of two straight lines along radius under $\hat{\mathbf{x}}$ direction at times $t = 0, L$, which are actually vanishing in virtue of gauge condition (1), and of two lines parallel to the time axis at radii $x = R', R''$. Hence there leaves from (2) just the contribution

$$\oint_\Gamma A_\mu dx^\mu = \int_0^L dt \, (A_0(t, R'\hat{\mathbf{x}}) - A_0(t, R''\hat{\mathbf{x}}) \,) = - \int_0^L dt \int_{R'}^{R''} dx \, E_\parallel (t, \mathbf{x}) \, . \tag{6}$$

To perform Gibbs averaging some more elaborated technique is needed. In [3, 5] there was proposed the method of the effective action denoted as W that is depending on collective variables λ, ν. One may connect the generating functionals of longitudinal strengths

$$Z[\zeta, \eta] = \int_\leftrightarrow \mathcal{D}\mathbf{A}_\perp \, \mathcal{D}\mathbf{E}_\perp \; \exp{(-)} \int_\Omega dx \, [\, \mathbf{E}\dot{\mathbf{A}} - \frac{1}{2}\mathbf{E}_{FS}^2 + \frac{1}{2}\mathbf{B}_{FS}^2 + i \, \zeta E_\parallel - \eta B_\parallel \,] \tag{7}$$

and of collective fields ($\lambda = -\partial_x \sigma$)

$$\mathcal{Z}[\zeta, \eta] = \int_\leftrightarrow \mathcal{D} \, \lambda \, \mathcal{D} \, \nu \;\; \exp{[-W[\lambda, \eta] - i \, (\, \lambda \, \zeta + \eta \, \nu \,)\,]} \tag{8}$$

via the relation

$$Z[\zeta, \eta] = \exp{[\, \frac{1}{2} \int_\Omega dx \; (\, \zeta^2 + \eta^2 \,) \,]} \;\, \mathcal{Z}[\zeta, \eta] \, . \tag{9}$$

Let $F(E_\parallel, B_\parallel)$ be an arbitrary functional of its arguments. Then its average

$$\langle \, F(E_\parallel, B_\parallel) \, \rangle = Z^{-1}[\zeta, \eta] \, F(\, i \, \frac{\delta}{\delta \, \zeta} \, , \, \frac{\delta}{\delta \, \eta} \,) \; Z[\zeta, \eta] \Big|_{\zeta = \eta = 0} \tag{10}$$

is expressed through the $\langle F(\lambda, \nu) \rangle$ of $\mathcal{Z}[\zeta, \eta]$'s by means of the formula

$$\begin{aligned} \langle \, F(\, E_\parallel \, , \, B_\parallel \,) \, \rangle = {} & \textstyle\int_\leftrightarrow \mathcal{D} \, \xi \, \mathcal{D} \, \theta \;\, \exp{(\, -\frac{1}{2} \int_\Omega dx \, [\, \xi^2 + \theta^2 \,] \,)} \\ & \langle \, F(\, \lambda + i \, \xi \, , \, -i \, \nu + \theta \,) \, \rangle \, . \end{aligned} \tag{11}$$

Here integrations must be carried out over fields from L^2-space. This relation lies in a heart of collective variables interpretation. They ought be imagined as longitudinal (chromo) electric and magnetic fields smoothed by a Gaussian noise.

Combining previous results we infer the basic formula

$$\mathcal{W}[\Gamma] = \int_{\leftrightarrow} \mathcal{D}\xi\, \mathcal{D}\sigma\, \mathcal{D}\nu\, e^{-W[\sigma,\nu]-\frac{1}{2}\int_\Omega dx\, \xi^2}\, \mathrm{tr}\, \mathrm{T}\, \exp\Big(ig \int_0^L dt \int_{R'}^{R''} dx\, [\lambda + i\,\xi](t,\mathbf{x}) \Big)\,. \tag{12}$$

Time was converted into Euclidean ($L \le \beta$). Dyson $\mathrm{T}\exp$ stands because coefficients fields depend upon time. Trace with an upper–case letter means a trace of operators in the Hilbert space not commuting at different instants. A lower–case trace in (12) is a Lie–algebra fundamental representation one.

Confined phase is characterized by insertion of singlet projector P_s of the group of *gauge transformations at infinity* G_∞ into observables' Gibbs averages

$$\langle F \cdots G \rangle^{conf} = \langle P_s F \cdots G \rangle. \tag{13}$$

We should take this operator, which acts in the Hilbert space accepting a unitary representation of G_∞, into consideration when the auxiliary variable λ is introduced. It is important to stress that the integral over group can not be permuted with path integration. This ensues from noncommutativity of generators Q^a with different indices. Therefore we may evaluate the group integral in the end of calculations or include the projector as $P_s \propto \delta[Q] \equiv \prod_{\hat{\mathbf{x}},a} \delta[Q^a(\hat{\mathbf{x}})]$ via appropriate delta–function inside path integral. The second possibility is a more suitable here

$$\begin{aligned} Z \;\; = \;\; & \int_{\leftrightarrow} \mathcal{D}A_\perp\, \mathcal{D}E_\perp\, e^{-W[A,E]}\, \delta[Q] = \int_{\leftrightarrow} \mathcal{D}\lambda\, \mathcal{D}A_\perp\, \mathcal{D}E_\perp\, e^{\frac{1}{2}\int_\Omega dx\,(E_\perp^2 - B^2)} \\ & e^{\int_\Omega dx\, [-\frac{1}{2}\lambda^2 + \lambda E_\parallel]}\, \delta[Q]\,. \end{aligned} \tag{14}$$

To be self–consistent we present our path integral in terms of product of usual integrals over Fourier–modes of the complete and orthonormal basis formed by the spherical waves

$$f_k(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin\, kx}{kx} \tag{15}$$

multiplied by spherical functions Y_{lm}. Introducing new variable φ related with λ via

$$\lambda = \frac{1}{x^2}\int_0^x y^2\, dy\ \varphi(y\hat{\mathbf{x}})\,. \tag{16}$$

we express the integral of interest in the following manner

$$\mathcal{I} = \int \prod_{n,lm} d\tilde{\varphi}_{n,lm}(k)\ e^{\sum_{n,lm}\int_0^\infty dk\,\left[-\frac{1}{2}|\,\tilde{\varphi}\,|^2_{n,lm}(k) + \tilde{\Phi}^*_{n,lm}(k)\,\tilde{\varphi}_{n,lm}(k)\right]}\,. \tag{17}$$

The confinement requirement implies then

$$\tilde{\Phi}_{n,lm}(k)\ |_{k=0} = 0\,, \tag{18}$$

that is equivalent up a normalization constant to taking integration over fields obeying $\tilde{\varphi}_{n,lm}(k=0)=0$. The latter leads us to further simplifications.

Parameters R', R'' are at our disposal. We can choose them to simplify evaluation a great deal. Namely let us arrange the limit of the kind $R', R'' \to \infty, \quad R''-R' \equiv \Delta R = const$. In (12), the integral

$$\int_{R'}^{R''} dx\, \lambda_{n,\,lm}(x) \to -\frac{\Delta R}{R'R''} 2^{3/2}\pi^{1/2}\, \tilde{\varphi}_{n,\,lm}(0) = 0 \tag{19}$$

tends to zero. So far we have to take just a simple integral over the auxiliary field ξ. For this purpose we first of all make the following remarks.

The generators of SU(N) fundamental representation are N dimensional matrices satisfying the conditions

$$\mathrm{tr}(\lambda^a\, \lambda^b) = 2\, \delta^{ab}\,, \qquad \mathrm{tr}\lambda^a = 0\,. \tag{20}$$

Coordinate frame would be conveniently to choose so that the matrices of $(N-1)$ commuting generators will be represented by diagonal matrices $\{\lambda^{\alpha}_{kk} \equiv \lambda^{(\alpha)}_k,\ (\alpha = 1,\ldots,N-1)\}$. Implementing gauge rotations all but $(N-1)$ of ξ's components in respect to these generators might be made zero:

$$\xi = \sum_{\alpha=1}^{N-1} \xi^{\alpha}\, \frac{\lambda^{\alpha}}{2\,i}\,. \tag{21}$$

The characteristic condition (20) regains formally in terms of $\vec{\lambda}^{(\alpha)}$ N- dimensional vectors after introduction of the vector

$$\lambda^{(0)} = \frac{1}{\sqrt{N}}\,(\underbrace{1,\ldots,1}_{N})\,, \tag{22}$$

as a requirement that the set $\left\{\lambda^{(0)}, \lambda^{(\alpha)}/\sqrt{2}\right\}$ is an orthonormal frame in N-dimensional vector space. The completeness of such basis presupposes unit decomposition

$$\sum_{\alpha=1}^{N-1} \lambda^{(\alpha)}_i\, \lambda^{(\alpha)}_j = 2\left(\delta_{ij} - \lambda^{(0)}_i\, \lambda^{(0)}_j\right). \tag{23}$$

These obvious remarks being done let us return to evaluation of our integral. One can prove that any normalized vector $\hat{\xi}(t)$ may be rotated to make it time–independent. Since then on the T-exponential becomes a usual one

$$J = \int_{\leftrightarrow} \mathcal{D}\vec{\xi}\, e^{-\frac{1}{2}\int d^4x\, \vec{\xi}^{\,2}}\ \mathrm{tr}\ \exp\left(-g\int_0^L dt \int_{R'}^{R''} dx\, \xi(t, x\hat{\mathbf{x}})\right). \tag{24}$$

Provided special basis (21) choice the factorization of multiples takes place

$$J = \int_{\leftrightarrow} \prod_{\alpha=1}^{N-1} \mathcal{D}\xi^{\alpha}\, e^{-\frac{1}{2}\int d^4x\, (\xi^{\alpha})^2} \sum_{i=1}^{N} \exp\left[\sum_{\alpha=1}^{N-1} (j \bullet \xi^{\alpha})\, \lambda^{(\alpha)}_i\right], \tag{25}$$

$$j(x) = \frac{i\,g}{2}\,\theta(0 \le t \le L)\,\frac{\theta(R' \le x \le R'')}{x^2}\,\delta(\hat{\mathbf{x}} - \hat{\mathbf{x}}') \tag{26}$$

giving rise to the result

$$J = \sum_{i=1}^{N} \exp\left[-\sum_{\alpha=1}^{N-1} (\lambda_i^{(\alpha)})^2\,\chi_0\,S\right] \tag{27}$$

and consequently,

$$\mathcal{W}_{SU(N)} = N\,\exp\left(-\chi_{SU(N)}\,S\right), \qquad S = L\,(R'' - R') \tag{28}$$

with the string tension

$$\chi_{SU(N)} = 2\left(1 - \frac{1}{N}\right)\chi_0\,. \tag{29}$$

We have taken into account formula (23) for the sum inside the exponential. This expression is nothing but the long–awaited area law with the SU(2) string tension

$$\chi_0 = \lim_{R', R'' \to \pi\,\delta(0)} \frac{g^2\,\delta(\hat{\mathbf{0}})}{8\,R'\,R''}\,. \tag{30}$$

Surely, the accurate definition of this limit needs some care. Both radii in a self–consistent derivation should be tended to the space infinity related with a more familiar infinity like $\delta(0)$ by

$$\delta(0) = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{2\pi}\,. \tag{31}$$

There are two proper way of understanding such infinities. First, everything being cut–offed and kept finite till the final result, as it conventionally was done, has no problems to cope with. Next, infinite objects above arise naturally enough in nonstandard analysis [9]. Fortunately, we mustn't deepen in those dark mathematical areas since the construction (30) has been already familiar for us if reliable formula (31) is borne in mind. Comparison with critical temperature of [10-14] leads to the relation

$$\chi_0 = \frac{\pi^2\,T_c^2}{a_c}\,, \qquad a_c \approx 2.61882\ldots \tag{32}$$

For the dimensionless ratio $\xi \equiv \frac{T_c}{\sqrt{\chi}}$ we predict in the case of SU(3) group the value

$$\xi_{SU(3)} = \frac{3}{2\sqrt{2}\,\pi}\sqrt{a_c} \approx 0.55\,, \tag{33}$$

whereas the lattice datum [10-14] is 0.58 ± 0.04. Another data are summarized in Table.

The visual picture of colour confinement following from the (13) is that *for any physical state contributing to the thermal average the flow of colour to*

relat.	lattice	theoretical
$\frac{T_c^{SU(3)}}{T_c^{SU(2)}}$	1.16 ± 0.07	$\sqrt{3/2} \simeq 1.22$
$\frac{\xi^{SU(3)}}{\xi^{SU(2)}}$	1.04 ± 0.18	$3/(2\sqrt{2}) \simeq 1.06$

Tab. 1: Comparison of our theoretical predictions with lattice data.

infinity equals to zero in every direction. Hence the coloured particles could not leak to spatial infinity and may no be observed.

Another possible insight on confinement is provided by Wilson loop. The matter of the present paper was to show how the first picture leads to the latter one. The results are not only positive but have also fair agreement with data of the lattice simulations [10-14].

We have unexpectedly disclosed that the combination of divergences constituting the only parameter of the gluodynamics effective potential possesses a direct physical sense. Such quantity is related with the string tension in the area law.

It is worthy to emphasize that the derivation of the area law was itself exact. The singletness under G_∞'s action has not been proved rigorously of course, though being taken for granted allows one to get such strong an assertion. The ratio ξ is known approximately as it depends on the critical temperature.

Results presented so far need a more accurate procedure of ultraviolet regularization, which is rather unclear in the noncovariant FS gauge. It is a good luck that the final expression for the ratio ξ is free of such problems. Let us make also some remarks about infrared regularization. Use of objects like $\delta(0)$ is not at any rate worse than use of Dirac delta function itself. Recently the procedure of space cut–off was carried out rigorously for Wilson loop and free energy density. The Dirichlet boundary condition imposed on the 3-d sphere allows to get the same results which are just the finite–volume repetition of our present formulae.

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