Dependence of the partition function on the boundary condition and confinement in the gauge field theories

N.A. Sveshnikov, E.G. Timoshenko*

Nuclear Physics Institute, Moscow State University, Moscow, 119899, Russia

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Abstract

We analyse dependence of the partition function on the boundary condition for the longitudinal component of the electric field strength $E_\parallel$ in the gauge field theories. In a physical gauge the Gauss law constraint may be resolved explicitly expressing this component via an integral of the physical transversal variables. Therefore the Dirichlet boundary condition $R^2 E_\parallel(R\hat{\mathbf{x}}) = \chi(\hat{\mathbf{x}})$ is in fact an additional constraint resulting in a nontrivial dependence of the partition function Z on χ in the thermodynamic limit. In particular, we consider quantum electrodynamics with an external charge and $SU(2)$ gluodynamics. We find that only charge distributions slowly decreasing at spatial infinity produce a nontrivial dependence in the Abelian theory. However, in gluodynamics for temperatures below some critical value the partition function acquires a delta–function like dependence on $\chi(\hat{\mathbf{x}})$, which leads to colour confinement.

*Permanent address: Department of Chemistry, University College Dublin, Belfield, Dublin 4, Ireland. E-mail: Edward.Timoshenko@ucd.ie

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It is with a feeling of deep sadness and irrecoverable loss that I had to complete writing this paper on my own. In the name of N.A. Sveshnikov Theoretical Physics has lost an extremely talented scientist, a great educator and a very nice person. No matter how big influence Nikita Alexeevich's ideas had on his students, one of which I am privileged to be, and on colleagues who knew him well, it would be only fair to say that it is for future generations to judge their true merit.

Perhaps, this paper may be viewed as a logical conclusion to years of our joint research on the problem of confinement. Although, in my view, it does shed light on one of the most difficult problems in Elementary Particles Physics, it is really only the beginning. For standing on a top of a high mountain one can better see how little we actually know. It was one of N.A. Sveshnikov's gifts to be able to see farther ahead in search for the hidden mathematical beauty of the physical world, the beauty that expresses the essence of all things.

E. T.

1 Introduction

There is a considerable tradition in the field theory (see e.g. Ref. [1]) to neglect any surface terms inevitably appearing in derivations. This is usually motivated by a fast decrease of all fields at spatial infinity. Such behaviour is only natural for theories with short–range interactions, but it is by no means obvious if long–range interactions come into play.

The same problem acquires a somewhat different form at finite temperature. Obviously, the partition function of a translationally invariant system is ill–defined in infinite volume. Therefore, initially one has to enclose the system into a finite domain V and to assume some boundary conditions at the boundary ∂V. Then, to analyse dependence on boundary conditions we have to consider the functional,

$$Z[\chi] = \mathrm{Tr}\left(e^{-\beta H_V}\,\delta(\phi|_{\partial V} - \chi)\right), \tag{1}$$

where H_V is the Hamiltonian of the system in volume V, β is the inverse temperature, ϕ is some subset of the canonical variables and function χ defined on the boundary ∂V specifies the Dirichlet boundary conditions for the latter. In this framework we shall call $Z[\chi]$ the *effective partition function*, its introduction being analogous in spirit to that of the effective action. It is reasonable to expect that $Z[\chi]$ and any thermodynamic function become independent of the particular choice of boundary conditions in the thermodynamic limit $V \to \infty$. Once again, one may doubt whether that would really be the case for long–range interacting systems.

It has been early recognised in the theory of gravity [2] that there are important physical situations in which boundary terms may have a physical meaning. We would like to mention also that many problems in hydrodynamics, such as e. g. description of surface waves [3], do require to consider variables at the boundary and nonvanishing surface terms.

Recently the interest to boundary effects in various field theories has been rapidly increased [4]. It has been found in the framework of the algebraic quantum field theory [5] that certain 2-dimensional models possess a nontrivial dynamics of the *variables at infinity*[1], and that such dynamics is responsible for the phenomenon of the dynamic mass generation.

Our current purpose is to emphasise the role of boundary terms in the 4-dimensional gauge field theory at finite temperature and to study physical effects they can produce. In this case the appearance of a nontrivial χ-dependence in $Z[\chi]$ in the thermodynamic limit is almost obvious for the following reasons. The gauge theory is initially formulated in terms of an enlarged set of variables, the vector potentials $\mathbf{A}$ and the electric field strengths $\mathbf{E}$ in the Hamiltonian formulation, that make the gauge invariance explicitly manifest. Further, to obtain observable quantities one has to project the theory onto a subset of the *physical* variables by resolving the Gauss law constraint, $\boldsymbol{\nabla E} = \rho$, and by adopting a gauge condition. Boundary conditions, of course, have to be compatible with these. For instance, in the Abelian case consider the boundary condition for the electric field strength component $E_\parallel$ normal to the boundary ∂V, which we take for simplicity as a sphere of radius R,

$$R^2\, E_\parallel(R\hat{\mathbf{x}})\,|_{\partial V} = \chi(\hat{\mathbf{x}}). \tag{2}$$

This variable then has to obey the integrated form of the Gauss law, and

[1]These are defined as limits of the boundary values in the weak topology.

therefore,

$$Z[\chi] \propto \delta(\int_{\partial V} d\hat{\mathbf{x}}\, \chi - \int_V d\mathbf{x}\, \rho). \tag{3}$$

We would like to emphasise that the dependence of the effective partition function on the boundary condition imposed on $E_\parallel$ is of primary importance in the gauge theory because of the direct relation of this component to the colour charge flux due to the Gauss law. Analysis of namely this dependence will be the matter of the current paper. Based on the knowledge of this dependence alone we can suggest a simple *confinement criterion*:

$$Z[\chi] \propto \prod_{\hat{\mathbf{x}}} \delta(\chi(\hat{\mathbf{x}})). \tag{4}$$

This condition simply means that the colour flux at the boundary is strictly zero in *every* spatial direction at infinity for any state belonging to the Hilbert space of the system.

The plan of the paper is as follows. In Sec. 2 we proceed with a careful calculation of of $Z[\chi]$ in the simplest case of the Abelian theory with an external charge density. Then, in Sec. 3 we reproduce the answer obtained in the previous section using another technique, which is also applicable to the non–Abelian theory. Sec. 4 is devoted to calculation of $Z[\chi]$ using the mean–field approximation and to consequent analysis of the confinement phase transition in gluodynamics.

2 Partition Function of QED with an External Charge

The partition function in a finite domain V may be represented by an Euclidean path integral over the fields periodic in time on the interval $[0, \beta]$, where β is the inverse temperature. The path integral is well–defined only if some boundary conditions are specified on all fields at the boundary ∂V.

Let us denote the transversal and longitudinal components of vectors with respect to the gradient ∂ as superscripts $\boldsymbol{\partial}\mathbf{A}^\perp = 0$, and with respect to the vector $\mathbf{x}$ as subscripts $\mathbf{x}\mathbf{A}_\perp = 0$. For simplicity we choose the Coulomb gauge, $\mathbf{A} = \mathbf{A}^\perp$. We note also that the condition $\mathbf{A} = \mathbf{A}_\perp$ corresponds to the Fock–Schwinger gauge [6]. We shall also assume that $V = \{\mathbf{x} : \ |\mathbf{x}| = R\}$

is a spherical domain with the radius R. The Gauss law constraint,

$$\boldsymbol{\partial}\mathbf{E} = \rho(\mathbf{x}) \tag{5}$$

allows to eliminate one space component of the electric field strength $\mathbf{E}$.

In a previous work [7] on the basis of the general result [8] we have developed the Hamiltonian formalism for the system in a finite spherical domain incorporating the boundary values as Hamiltonian variables. We have shown that the boundary conditions $\mathbf{E}_\perp(R\hat{\mathbf{x}}) = 0$ and $x_j F_{ij}(R\hat{\mathbf{x}}) = 0$ are consistent with the localised time evolution in the Fock–Schwinger gauge. By transforming the theory to the Coulomb gauge one would arrive instead at the boundary condition of the form $\mathbf{E}^\perp(R\hat{\mathbf{x}}) = 0$. Since these variables are independent, the dependence on a particular choice of the boundary conditions disappears for infinite system. The situation is quite different for the component $E_\parallel(R\hat{\mathbf{x}}) = \hat{\mathbf{x}}\mathbf{E}(R\hat{\mathbf{x}})$. Indeed, Eq. (5) may be easily solved,

$$E_\parallel(\mathbf{x}) = \frac{1}{x^2}\int_0^x y^2 dy\,(\rho - \boldsymbol{\partial}\mathbf{E}_\perp)(y\hat{\mathbf{x}})\,. \tag{6}$$

Both types of the transversal variables are connected by the relation, [9]

$$\mathbf{E}_\perp(\mathbf{x}) = \mathbf{E}^\perp(\mathbf{x}) - \boldsymbol{\partial}\int_0^x dy\,\hat{\mathbf{x}}\mathbf{E}^\perp(y\hat{\mathbf{x}})\,. \tag{7}$$

Combination of Eqs. (6) and (7) now yields,

$$R^2 E_\parallel(R\hat{\mathbf{x}}) - \int_0^R y^2 dy\,\rho(y\hat{\mathbf{x}}) = \hat{\Delta}\int_0^R (R-y)dy\,\hat{\mathbf{x}}\mathbf{E}^\perp(y\hat{\mathbf{x}})\,, \tag{8}$$

where we have used the spherical part of the Laplacian $\Delta \equiv x^{-1}\partial_x^2\, x + x^{-2}\hat{\Delta}$. It is clear that the requirement $E_\parallel(R\hat{\mathbf{x}}) = \chi(\hat{\mathbf{x}})$, where $\chi(\hat{\mathbf{x}})$ is arbitrary, is nothing but a constraint on the physical variables $\mathbf{E}^\perp$. As we have seen [7], it is this constraint that makes the finite volume Hamiltonian formalism closed.

Now then, the partition function of QED with an external charge may be represented by the following path integral in the Coulomb gauge,

$$\begin{aligned} Z = &\int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{E}\;\delta(\boldsymbol{\partial}\mathbf{A})\,\delta(R^2 E_\parallel(R\hat{\mathbf{x}}) - \chi(\hat{\mathbf{x}})) \\ &\exp\int_\Lambda d^4x\left(i\,\mathbf{E}\dot{\mathbf{A}} - \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{A}\Delta\mathbf{A} - \frac{1}{2\epsilon^2}(\boldsymbol{\partial}\mathbf{E} - \rho)^2\right)\,, \end{aligned} \tag{9}$$

where we have used the notation for the domain $\Lambda = [0,\beta]\times V$ (we shall also use the notation $\partial\Lambda = [0,\beta]\times\partial V$). We have also introduced a regularisation of the Gauss law by the parameter ϵ, which should be set equal to zero at the end of calculations. This Gaussian integral is evaluated in a standard manner by a shift of the integration variables. To find the integral over $\mathbf{E}$ we introduce a new integration variable, $\mathbf{E}_1$,

$$\mathbf{E} = \mathbf{E}_1 + \boldsymbol{\mathcal{E}}\,, \qquad \mathbf{E}_1(R\hat{\mathbf{x}}) = 0\,. \tag{10}$$

Here the new variable $\mathbf{E}_1$ satisfies the zero boundary condition and $\boldsymbol{\mathcal{E}}$ is chosen so that there is no linear term in $\mathbf{E}_1$. This gives the equation on $\boldsymbol{\mathcal{E}}$,

$$i\dot{\mathbf{A}} - \boldsymbol{\mathcal{E}} + \frac{1}{\epsilon^2}\boldsymbol{\partial}(\boldsymbol{\partial}\boldsymbol{\mathcal{E}}) - \frac{1}{\epsilon^2}\boldsymbol{\partial}\rho = 0\,, \tag{11}$$

$$R^2\hat{\mathbf{x}}\boldsymbol{\mathcal{E}}(R\hat{\mathbf{x}}) = \chi(\hat{\mathbf{x}})\,. \tag{12}$$

The latter boundary condition follows from the second delta–function in Eq. (9). We may decompose this vector onto the transversal and longitudinal parts in the momentum space, $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{\perp} - \boldsymbol{\partial}\varphi$. Then, the transversal part is simply $\boldsymbol{\mathcal{E}}^{\perp} = i\dot{\mathbf{A}}$, and the equation for φ becomes,

$$(\Delta - \epsilon^2)\varphi = -\rho\,, \tag{13}$$

$$R^2\frac{\partial\varphi}{\partial R} = -\chi(\hat{\mathbf{x}})\,. \tag{14}$$

The partition function (9) is further decomposed as the product,

$$Z = Z_1\tilde{Z}\,,$$

$$Z_1 = \int \mathcal{D}\mathbf{A}^{\perp}\mathcal{D}\mathbf{E}\,\exp\int_\Lambda d^4x\,\left(-\frac{1}{2}\dot{\mathbf{A}}^2 + \frac{1}{2}\mathbf{A}\Delta\mathbf{A} - \frac{1}{2}\mathbf{E}_1^2 - \frac{1}{2\epsilon^2}(\boldsymbol{\partial}\mathbf{E}_1)^2\right) \tag{15}$$

$$\tilde{Z} = \exp\beta\left(\frac{1}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\varphi(R\hat{\mathbf{x}}) - \frac{1}{2}\int_{\partial V} d\mathbf{x}\,\rho(\mathbf{x})\varphi(\mathbf{x})\right)\,, \tag{16}$$

where φ is the solution of Eqs. (13,14).

The solution of Eqs. (13,14) is, obviously, the sum of the homogeneous part, ϕ, satisfying the nontrivial boundary condition, and of the inhomogeneous part satisfying the zero boundary condition,

$$\varphi = \phi - G\bullet\rho \qquad G = (\Delta - \epsilon^2)^{-1}\,, \tag{17}$$

where G is the Green function corresponding to the zero Neumann boundary condition at R. The effective partition function can be presented as,

$$\begin{aligned}
\tilde{Z} &= \tilde{Z}_\chi \tilde{Z}_{\rho\rho} \tilde{Z}_{\rho\chi}\,, \\
\tilde{Z}_\chi &= \exp\left(\frac{\beta}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\phi(\hat{\mathbf{x}})\right)\,, \qquad (18)\\
\tilde{Z}_{\rho\rho} &= \exp\left(\frac{\beta}{2}\int_{\partial V} d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\right)\,, \qquad (19)\\
\tilde{Z}_{\rho\chi} &= \exp\left(-\frac{\beta}{2}\int_{\partial V} d\mathbf{x}\,\phi(\mathbf{x})\rho(\mathbf{x})\right)\exp\left(-\frac{\beta}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\,(G\bullet\rho)(\hat{\mathbf{x}})\right) \qquad (20)
\end{aligned}$$

It is natural to consider the problem further in terms of the spherical coordinates. Solution of Eq. (13) regular inside the sphere is given by,

$$\phi_{lm} = C_{lm}\sqrt{\frac{\pi}{2\epsilon r}} I_{l+1/2}(\epsilon r)\,, \qquad (21)$$

$$I_{n-1/2}(z) = \sqrt{\frac{2}{\pi z}} z^n \left(\frac{1}{z}\frac{d}{dz}\right)^n \cosh z\,. \qquad (22)$$

The constant C_{lm} is determined from Eq. (14). The role of the regulator ϵ now becomes clear. The zero mode solution is simply,

$$\phi_{00} = C_{00}\frac{\sinh\epsilon r}{\epsilon r}\,, \qquad C_{00} = -\frac{\chi_{00}}{R^2\epsilon(\cosh\epsilon R/\epsilon R - \sinh\epsilon R/(\epsilon R)^2)}\,, \qquad (23)$$

and it is $1/\epsilon^2$ singular as ϵ tends to zero. At the same time, solutions for other modes are perfectly regular in this limit and tend to,

$$\phi_{lm} = C_{lm}\, r^l\,, \qquad C_{lm} = -\frac{\chi_{lm}}{lR^{l+1}}\,. \qquad (24)$$

In consideration of the zero mode one must therefore be more careful and keep ϵ nonvanishing. The zero mode Green function, that is defined by,

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \epsilon^2\right) G_{00}(r,r') = \frac{\delta(r-r')}{rr'}\,, \qquad (25)$$

$$\left.\frac{\partial G_{00}(r,r')}{\partial r}\right|_{r=R} = 0\,, \qquad (26)$$

is easily calculated,

$$\begin{aligned} G_{00}(r,r') &= \frac{1}{\epsilon r r'}\Big(\frac{1}{2}\sinh\epsilon|r-r'| - \frac{1}{2}\sinh\epsilon(r+r') \\ &+ \frac{\sinh\epsilon R - \cosh\epsilon R/\epsilon R}{\cosh\epsilon R - \sinh\epsilon R/\epsilon R}\sinh\epsilon r\sinh\epsilon r'\Big)\,. \end{aligned} \tag{27}$$

The leading terms at small ϵ are,

$$G_{00}(r,r') \simeq -\frac{3}{\epsilon^2 R^3} + \frac{9}{5R} - \frac{1}{2}\frac{r^2+r'^2}{R^3} - \frac{1}{\max(r,r')}\,, \tag{28}$$

$$\phi_{00} \simeq -\left(\frac{3}{\epsilon^2 R^3} - \frac{3}{10R} + \frac{r^2}{2R^3}\right)\chi_{00}\,. \tag{29}$$

Substitution of these results into formulae (18-20) gives for the nonzero modes in the limit $\epsilon = 0$,

$$\tilde{Z}_{\chi\, l>0} = \exp\left(-\frac{\beta}{2R}\sum_{lm>0}\frac{|\chi_{lm}|^2}{l}\right)\,. \tag{30}$$

For simplicity, we assume that the distribution of the charge ρ is spherically symmetrical. In this case only the zero mode term survives. If we introduce the charge density momenta,

$$\mathcal{Q}_{00} = \int_0^R r^2\,dr\,\rho_{00}(r)\,, \qquad \mathcal{G}_{00} = \int_0^R r^4\,dr\,\rho_{00}(r)\,, \tag{31}$$

our results may be summarised as follows,

$$\begin{aligned} \tilde{Z}_{00} &= \exp\beta\Big(-\frac{1}{2}\int_0^R x^2dx\,y^2dy\,\frac{\rho_{00}(x)\,\rho_{00}(y)}{\max(x,y)} - \frac{3}{2\epsilon^2R^3}(\mathcal{Q}_{00}-\chi_{00})^2 \\ &+ \frac{1}{R}(\frac{9}{10}\mathcal{Q}_{00}^2 - \frac{1}{10}\chi_{00}^2 - \frac{3}{10}\chi_{00}\mathcal{Q}_{00}) - \frac{1}{2R^3}(\mathcal{Q}_{00}-\chi_{00})\mathcal{G}_{00}\Big)\,. \end{aligned} \tag{32}$$

In the limit $\epsilon \to 0$ this functional contains the delta–function of the condition $\mathcal{Q}_{00} = \chi_{00}$, and in addition we find the following correction to the standard answer due to the surface terms,

$$\tilde{Z}_{00} = \exp\left(\frac{\beta}{2R}\mathcal{Q}_{00}^2\right)\,\delta(\mathcal{Q}_{00}-\chi_{00})\,. \tag{33}$$

As a simple illustration of the above result it is instructive to consider the charge density $\rho_{00} = -\kappa/(\sqrt{\pi}\, r)$ corresponding to a linear rising electric potential $\varphi = \kappa\, r$. For such an exotic charge distribution, modeling the confinement alike potential, we find the additional constant contribution to the free energy density,

$$\Delta\mathcal{F} = -\frac{\log Z}{\beta V} = -\frac{3}{32\pi^2}\kappa^2\,. \tag{34}$$

It is interesting to note that this correction makes the free energy density smaller, and in this sense the boundary effects are thermodynamically significant. This example exhibits a promising connection between the boundary effects and the confinement phenomenon.

3 Collective variable formulation

In the present section we give a different formulation of the same problem by introducing the collective variable, σ, conjugate to the Gauss law constraint. Both formulations are completely equivalent in the Abelian theory, the transformation between them being just a trivial change of variables. However, the new formulation appears to be more fruitful in the non–Abelian gauge theory. Let us rewrite formula (9) in terms of the collective variable σ, introduced by the definition,

$$\exp\left(-\frac{1}{2\epsilon^2}\int_\Lambda d^4x\,(\boldsymbol{\partial}\mathbf{E}-\rho)^2\right) = \int \mathcal{D}\sigma\,\exp\int_\Lambda d^4x\,\left(-\frac{\epsilon^2}{2}\sigma^2 + i\sigma(\boldsymbol{\partial}\mathbf{E}-\rho)\right). \tag{35}$$

One starts by taking the integral over $\mathbf{E}$,

$$J = \int \mathcal{D}\mathbf{E}\,\exp\int_\Lambda d^4x\,\left(-\frac{1}{2}\mathbf{E}^2 + i\mathbf{E}\dot{\mathbf{A}} + i\sigma\boldsymbol{\partial}\mathbf{E}\right)\,\delta(R^2E_\parallel(R\hat{\mathbf{x}}) - \chi(\hat{\mathbf{x}})). \tag{36}$$

This can be done in analogy with the previous section by applying a shift $\mathbf{E} = \mathbf{E}_1 + \boldsymbol{\mathcal{E}}$ and using the decomposition $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^\perp - \boldsymbol{\partial}\varphi$. Then, we shall get: $\boldsymbol{\mathcal{E}}^\perp = i\dot{\mathbf{A}}$ and $\varphi = i\sigma$. Using the delta–function in the above formula one may rewrite the boundary term as $i\int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\,\chi\sigma$. In the term $\int d\mathbf{x}\,\dot{\mathbf{A}}\boldsymbol{\partial}\sigma$ the appropriate boundary term vanishes by the gauge condition. Thus, we obtain,

$$J = \exp\left(-\frac{1}{2}\int_\Lambda d^4x(\dot{\mathbf{A}}^2 + (\boldsymbol{\partial}\sigma)^2) + i\int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\,\chi\sigma\right)\,\delta(iR^2\frac{\partial\sigma}{\partial R} + \chi). \tag{37}$$

At this stage the χ dependence is contained only in the integral,

$$\begin{aligned}\tilde{Z} &= \int \mathcal{D}\sigma \, \exp\Big(- \int_\Lambda d^4x \, (\frac{1}{2}(\boldsymbol{\partial}\sigma)^2 + \frac{\epsilon^2}{2}\sigma^2 + i\sigma\rho) \\ &+ i \int_{\partial\Lambda} dt \, d\hat{\mathbf{x}} \, \chi\sigma \Big) \, \delta(iR^2 \frac{\partial\sigma}{\partial R} + \chi) \, . \end{aligned} \tag{38}$$

Such integral is evaluated straightforwardly by a shift of the integration variable, $\sigma = \sigma_1 + \varsigma$. The new variable satisfies the trivial boundary condition, $R^2\partial\sigma_1(R\hat{\mathbf{x}})/\partial R = 0$, and ς is found by the requirement that there is no term linear in σ_1 after the shift. This gives the equations on ς,

$$(\Delta - \epsilon^2)\varsigma = i\rho \, , \quad R^2 \frac{\partial\varsigma}{\partial R} = i\chi. \tag{39}$$

Using the boundary condition on ς we finally derive,

$$\tilde{Z} = \exp\left(-\frac{i}{2}\beta \int_V d\mathbf{x} \, \varsigma\rho + \frac{i}{2}\beta \int_V d\hat{\mathbf{x}} \, \varsigma \, \chi \right) \, . \tag{40}$$

Note that with the identification $\varsigma = -i\varphi$ we have precisely reproduced the answer of Eqs. (13,14,16).

4 Non–Abelian theory

In a previous paper [9] we have derived a representation for the partition function Z of SU(N) gluodynamics in a finite volume as the path integral over the collective variables analogous to that of the previous section. In order to find the effective action $Z[\chi]$ dependence we have to return to the beginning of that derivation in the Fock–Schwinger gauge,

$$\begin{aligned} Z[\chi] &= \int \mathcal{D}\mathbf{A} \, \mathcal{D}\mathbf{E} \, \mathcal{D}\sigma \, \exp\Big(\int_\Lambda d^4x (i \, \mathbf{E}\dot{\mathbf{A}} - \frac{1}{2}\mathbf{E}^2 - \frac{1}{2}\mathbf{B}^2 \\ &+ i \, \sigma \boldsymbol{\nabla}\mathbf{E}) \Big) \, \delta(A_\parallel) \, \delta(R^2 \, E_\parallel(R\hat{\mathbf{x}}) - \chi), \end{aligned} \tag{41}$$

where σ, clearly, is just a different notation for the temporal component of the gauge field, A_0. Obviously, any dependence on χ is concentrated in the path integral over $E_\parallel$,

$$I = \int \mathcal{D}E_\parallel \exp\Big(\int_\Lambda d^4x (-\frac{1}{2}E_\parallel^2 + i \, \sigma \, (\boldsymbol{\partial}\hat{\mathbf{x}})E_\parallel) \Big) \, \delta(R^2 E_\parallel - \chi). \tag{42}$$

This is calculated by a shift $E_\parallel = E_\parallel^1 + \mathcal{E}$, where $\mathcal{E} = -i\,\partial\sigma/\partial x$ and $E_\parallel^1$ satisfies the zero Dirichlet boundary condition. Such a derivation gives,

$$I = \exp\Big(\int_0^\beta dt\Big[-\frac{1}{2}\int_V d\mathbf{x}\Big(\frac{\partial\sigma}{\partial x}\Big)^2 + i\int d\hat{\mathbf{x}}\sigma\,\chi\Big]\Big)\,\delta(R^2\frac{\partial\sigma}{\partial R} - i\chi). \tag{43}$$

Next, by introducing the path integral representation,

$$\exp\Big(-\frac{1}{2}\int_\Lambda d^4x\,B_\parallel^2\Big) = \int \mathcal{D}\nu\,\exp\Big(\int_\Lambda d^4x\,\Big(-\frac{1}{2}\nu^2 + i\,\nu\,B_\parallel\Big)\Big), \tag{44}$$

and after performing the integrations over $\mathbf{A}_\perp$ and $\mathbf{E}_\perp$ (see Ref. [9] for more details) we finally obtain,

$$\begin{aligned}
Z[\chi] &= \int \mathcal{D}\sigma\,\mathcal{D}\nu\,\exp(-W[\sigma,\nu] + i\int_{\partial\Lambda} dt\,d\hat{\mathbf{x}}\,\sigma\chi)\,\delta(R^2\sigma' - i\chi)\,,\\
2W[\sigma,\nu] &= \nu\bullet\nu + \boldsymbol{\partial}\sigma\bullet\boldsymbol{\partial}\sigma + \boldsymbol{K}_-\bullet C_+^{-1}\bullet\boldsymbol{K}_+\\
&+ \boldsymbol{K}_+\bullet C_-^{-1}\bullet\boldsymbol{K}_- + \mathrm{tr}\log C_+C_-\,,\\
C_\pm &= -\Delta_x - \nabla_t^2 \pm D\,,\quad \boldsymbol{K}_\pm = \boldsymbol{\partial}_\pm\nu \pm \nabla_t\boldsymbol{\partial}_\pm\sigma\,,\\
\nabla_t^{ab} &= \delta^{ab}\partial_t - gt^{abc}\,\sigma^c\,,\quad D^{ab} = gt^{abc}\,\nu^c\,,
\end{aligned} \tag{45}$$

where the projected derivatives are defined by,

$$\partial_\pm^i = \Pi_\pm^{ij}\,\partial_j\,,\quad \Pi_\pm^{ij} = \frac{1}{2}(\delta^{ij} - \hat{x}^i\hat{x}^j \pm i\epsilon^{ijk}\hat{x}^k) \tag{46}$$

and the bullet denotes the 4-d integration over the domain Λ.

In the saddle point approximation one expands the action near the saddle point,

$$\begin{aligned}
W[\varsigma+\sigma_1] &= W[\varsigma] + \int_\Lambda d^4x\,\frac{\delta W}{\delta\varsigma(\mathbf{x})}\sigma_1(\mathbf{x})\\
&+ \int_{\partial\Lambda} dt\,R^2 d\hat{\mathbf{x}}\,\mathcal{E}^{(1)}[\varsigma]\,\sigma_1(R\hat{\mathbf{x}}) + \ldots.
\end{aligned} \tag{47}$$

In the zero (mean–field) approximation we may write,

$$\tilde{Z}[\chi] = \exp\Big(-W[\varsigma] + i\int_{\partial\Lambda} dt\,d\hat{\mathbf{x}}\,\varsigma\chi\Big)\,, \tag{48}$$

$$\frac{\delta\,W}{\delta\varsigma} = 0\,,\qquad R^2\frac{\partial\varsigma(R\hat{\mathbf{x}})}{\partial R} = i\chi, \tag{49}$$

where the contribution from the first Euler derivative of the action $\mathcal{E}^{(1)}[\varsigma]$ is precisely canceled with that from the surface term in Eq. (45).

Thus, in the mean–field approximation the dependence $Z[\chi]$ is controlled by the saddle point solution ς. As we have seen in the previous section, the Abelian gauge theory possesses only the trivial solution $\varsigma = 0$.

In Ref. [9] we have studied constant solutions of the mean–field equations. Let us reproduce those results briefly here, but in addition carefully keeping a finite volume. For simplicity we also restrict ourselves to the gauge group $SU(2)$. First of all, we can introduce the notations for the free energy density, $\mathcal{F}_R$,

$$W_R = \beta V_R \mathcal{F}_R\,, \quad \mathcal{F}_R = \gamma_R F_R\,, \quad \gamma_R = \frac{8\pi^2 R\delta(\hat{0})}{\beta^2 V_R}\,, \tag{50}$$

where $V_R = 4\pi R^3/3$ is the domain volume and $\delta(\hat{0}) = \frac{1}{4\pi}\sum_l (2l+1)$ is the ultravioletly divergent angular delta–function with coinciding arguments. The function F_R is now expressed via the dimensionless variables,

$$\sigma = \frac{2\pi\, s}{\beta\, g}, \qquad \nu = i\left(\frac{2\pi\, u}{\beta\, g}\right)^2\,, \tag{51}$$

where to produce a real mean magnetic field ν has to be purely imaginary (see Eq. (44)). After introducing the control parameter $a = (2\pi)^4/(2g^2\beta^4\gamma_R)$ and carrying out some derivations we obtain,

$$F_R[u,s] = -a\, u^4 + \mathcal{U}_R[u,s]\,, \tag{52}$$

$$\mathcal{U}_R[u,s] = \mathcal{U}_R[s] + \mathcal{V}_R[u,s]\,, \tag{53}$$

$$\mathcal{V}_R[u,s] = \frac{\beta}{2\pi R}\sum_{n=-\infty}^{\infty} \log\frac{L_R((n+s)^2+u^2)\, L_R((n+s)^2-u^2)}{L_R^2((n+s)^2)}\,, \tag{54}$$

$$\mathcal{U}_R[s] = \frac{\beta}{\pi R}\sum_{m=0}^{\infty} \log\left(1 - \frac{\cos 2\pi s}{\cosh(\pi(m+1/2)\beta/R)}\right)\,, \tag{55}$$

where $L_R(x) = \cosh(2\pi R\sqrt{x}/\beta)$.

It can be seen that at finite R this free energy possesses only a trivial minimum at $s = u = 0$. The situation changes after taking the thermodynamic limit, $R \to \infty$. The resulting expression for the free energy density (see Ref. [9]) possesses only the trivial stable solution at high temperatures $u = s = 0$. However, at some critical temperature the system undergoes a

first order phase transition, below which there appears a deeper nontrivial minimum at $u = s = 1/2$ (see Ref. [9]).

Therefore, at high temperatures $T > T_c$ the dependence $Z[\chi]$ is determined by the solution of the linearised equation $\delta^2 W/\delta\varsigma^2 \bullet \sigma_1$ around $\varsigma = 0$. This can only produce the dependence akin to the Abelian theory. Namely, it contains the delta–function expressing the conservation of the global charge (Eq. (3) with $\rho = 0$), and apart from that it is trivial in the sense that $Z[\chi_{lm}] \to 1$ in the thermodynamic limit $R \to \infty$. This situation, obviously, corresponds to the *deconfinement* phase as there is no restriction on the colour fluxes at infinity.

However, below the critical temperature $T < T_c$ there is a nonzero constant solution $|\varsigma| = \pi/g\beta$. Since the system in invariant under the group of the *big gauge transformations* G_∞ parametrised by matrices $U(\hat{\mathbf{x}})$, the corresponding unit colour vector $\hat{\boldsymbol{\varsigma}}(\hat{\mathbf{x}})$ is arbitrary in every direction $\hat{\mathbf{x}}$. After integration over the orbits of the group SU(2) at each cone $\hat{\mathbf{x}}$ the dependence becomes,

$$\tilde{Z}[\chi] \;=\; \prod_{\hat{\mathbf{x}}} \frac{\sin \Delta\beta\, \varsigma\chi(\hat{\mathbf{x}})}{\Delta\beta\, \varsigma\chi(\hat{\mathbf{x}})} \sim \exp\left(-\frac{\Delta\pi^2}{g^2} \int_{\partial V} d\hat{\mathbf{x}}\, \chi^2(\hat{\mathbf{x}})\right) \tag{56}$$

where we have introduced a discretisation of the unit sphere with Δ being the infinitesimal cone area. It is well known [1] that in the continuous limit the bare coupling constant vanishes $g \to 0$ thereby making $Z[\chi]$ a very sharply peaked function around the zero argument. This corresponds to the *confinement* phase, in which colour fluxes are equal to zero in every spatial direction at infinity.

So, we can conclude that the dependence of the effective free energy on χ is the following,

$$Z[\chi] = \begin{cases} \prod_{\hat{\mathbf{x}}} \delta(\chi(\hat{\mathbf{x}})), & T < T_c, \\ 1, & T > T_c. \end{cases} \tag{57}$$

Therefore, the Gibbs average of an observable A is given by,

$$< A >= \begin{cases} \langle A \rangle_0, & T < T_c, \\ \int d\chi(\hat{\mathbf{x}})\, \langle A \rangle_\chi, & T > T_c, \end{cases} \tag{58}$$

where we have introduced the averages over "pure" states,

$$\langle A \rangle_\chi = \frac{1}{Z[\chi]} \int \mathcal{D}\sigma\, \mathcal{D}\nu\, e^{-W[\sigma,\nu] + i \int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\, \sigma\chi}\, A[\sigma,\nu]\,. \tag{59}$$

It is straightforward to see that any Gibbs average at low temperatures contains the singlet projector of the group G_∞,

$$\langle A\rangle_0 = \lim_{R\to\infty}\frac{1}{Z_R[0]}\mathrm{Tr}\,(e^{-\beta H_R}\,\delta(Q_R)\,A) = \lim_{R\to\infty}\frac{1}{Z_R[0]}\mathrm{Tr}\,(e^{-\beta H_R}\,P_s\,A)\,, \quad (60)$$

where $Q_R = \int_{V_R} d\mathbf{x}\,\boldsymbol{\nabla}\boldsymbol{E}$ is the operator of the colour charge in volume V and P_s is the singlet projector of the big gauge transformations. Presence of this projector in the Gibbs averages has been demonstrated to lead to the area law for the Wilson loop [10, 9], what is considered to be a standard confinement criterion.

5 Conclusion

The Gauss law in the gauge field theory may be resolved explicitly in a physical gauge. This produces effectively non–local interactions generating a boundary nontriviality of the theory.

We have studied the dependence of the effective partition function on the Dirichlet boundary condition $R^2E_\parallel(R\hat{\mathbf{x}}) = \chi(\hat{\mathbf{x}})$ imposed on the residual component of the electric field for electrodynamics with an external charge and SU(2) gluodynamics. In the Abelian case this dependence always contains the delta function expressing the conservation of the total charge, and it is nontrivial only for charge distributions slowly decreasing at spatial infinity.

The non–Abelian self–interactions lead to a more unusual effect. Here the restriction of possible boundary values of the longitudinal electric field at low temperatures provides the confinement mechanism proposed by us in Ref. [9]. Indeed, this quantity is proportional to the flux of the electric field through an infinitesimal cone in the direction $\hat{\mathbf{x}}$ at infinity. Therefore, since the colour flux vanishes for any direction, no colour could escape to infinity and be experimentally observed.

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